

# DUALITY FOR SYMMETRIC HARDY SPACES OF NONCOMMUTATIVE MARTINGALES

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**ABSTRACT.** We show relationship between the symmetric Hardy space of noncommutative martingales and its conditioned version. We derive a generalization of Burkholder-Rosenthal inequality to noncommutative symmetric spaces.

## 0. INTRODUCTION

The theory of noncommutative martingale inequalities has been rapidly developed. Many of the classical martingale inequalities have been transferred to the noncommutative setting. Here we mention only three of them directly related with the objective of this paper. The first one is the noncommutative Burkholder-Gundy inequality. In the fundamental work [26], Pisier/Xu established the noncommutative Burkholder-Gundy inequalities of noncommutative martingales in noncommutative  $L_p$ -spaces. In [12], the authors proved Burkholder-Gundy inequalities for martingale difference sequences in certain noncommutative symmetric spaces. Some versions can be found in [2, 3, 9].

The second one is the noncommutative Burkholder-Rosenthal inequality. Junge/Xu [18, 19] obtained the noncommutative analogue of the classical Burkholder/Rosenthal inequalities on the conditioned (or little) square function in noncommutative  $L_p$ -spaces. In [12], the authors gave a generalization of Rosenthal inequality to noncommutative symmetric spaces. Dirksen [10] proved that if symmetric space's Boyd index satisfy  $2 < p_E \leq q_E < \infty$ , then the Burkholder-Rosenthal inequalities hold in corresponding noncommutative symmetric space.

The third result is that independently Junge/Mei[17] and Perrin[25] obtained the following relationship between Hardy space of noncommutative martingales and its conditioned version:

$$H_p^c(\mathcal{M}) = h_p^c(\mathcal{M}) + h_p^d(\mathcal{M})$$

for all  $1 \leq p \leq 2$ .

The purpose of this paper is to extend the above relationship to the noncommutative symmetric case. We use this result and Burkholder-Gundy inequalities to derive a generalization of Burkholder-Rosenthal inequalities to noncommutative symmetric spaces. The main novelty of our paper is the following duality for conditional Hardy spaces: If  $E$  is a separable symmetric space with  $1 < p_E \leq q_E < 2$  then

$$(h_E^c(\mathcal{M}))^* = h_{E^\times}^c(\mathcal{M}), \quad (h_E^r(\mathcal{M}))^* = h_{E^\times}^r(\mathcal{M}).$$

As applications of this result, we obtain the noncommutative generalization of Burkholder-Rosenthal inequalities:

$$(0.1) \quad L_E(\mathcal{M}) = h_E^c(\mathcal{M}) + h_E^d(\mathcal{M}) + h_E^r(\mathcal{M})$$

for separable symmetric space  $E$  satisfying  $1 < p_E \leq q_E < 2$ .

Notice that (0.1) was independently proved recently in [27] under slightly different conditions via a very different argument.

The organization of the paper is as follows. In Section 1, we give some preliminaries and notations on noncommutative martingales and the noncommutative Hardy spaces. We prove the main result in Section 2.

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## 1. PRELIMINARIES

Now let  $E$  be a quasi-Banach lattice. Let  $0 < \alpha < \infty$ .  $E$  is said to be  $\alpha$ -convex (resp.  $\alpha$ -concave) if there exists a constant  $C > 0$  such that for all finite sequence  $(x_n)$  in  $E$

$$\begin{aligned} \|(\sum |x_n|^\alpha)^{\frac{1}{\alpha}}\|_E &\leq C(\sum \|x_n\|_E^\alpha)^{\frac{1}{\alpha}}, \\ (\text{resp. } \|(\sum |x_n|^\alpha)^{\frac{1}{\alpha}}\|_E &\geq C^{-1}(\sum \|x_n\|_E^\alpha)^{\frac{1}{\alpha}}). \end{aligned}$$

The least such constant  $C$  is called the  $\alpha$ -convexity (resp.  $\alpha$ -concavity) constant of  $E$  and is denoted by  $M^{(\alpha)}(E)$  (resp.  $M_{(\alpha)}(E)$ ). For  $0 < p < \infty$ ,  $E^{(p)}$  will denote the quasi-Banach lattice defined by

$$E^{(p)} = \{x : |x|^p \in E\},$$

equipped with the quasi-norm

$$\|x\|_{E^{(p)}} = \| |x|^p \|_E^{\frac{1}{p}}.$$

It is easy to check that if  $E$  is  $\alpha$ -convex and  $\beta$ -concave then  $E^{(p)}$  is  $p\alpha$ -convex and  $p\beta$ -concave with  $M^{(p\alpha)}(E^{(p)}) = M^{(\alpha)}(E)$  and  $M_{(p\beta)}(E^{(p)}) = M_{(\beta)}(E)$ . Therefore, if  $E$  is  $\alpha$ -convex then  $E^{(\frac{1}{\alpha})}$  is 1-convex, so it can be renormed as a Banach lattice (cf. [23], p. 54).

Let  $([0,1], \Sigma, m)$  be the Lebesgue measure space and  $L_0[0,1]$  be the space of all classes of Lebesgue measurable real-valued functions defined on  $[0,1]$ . Let  $x \in L_0[0,1]$ . Recall that the distribution function of  $x$  is defined by

$$\lambda_s(x) = m(\{\omega \in [0,1] : |x(\omega)| > s\}), \quad s > 0$$

and its decreasing rearrangement by

$$\mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\}, \quad t > 0.$$

For  $x, y \in L_0[0,1]$  we say  $x$  is submajorized by  $y$ , and write  $x \preceq y$ , if

$$\int_0^t \mu_s(x) ds \leq \int_0^t \mu_s(y) ds, \quad \text{for all } t > 0.$$

By a symmetric quasi Banach space on  $[0,1]$  we mean a Banach lattice  $E$  of measurable functions on  $[0,1]$  satisfying the following properties: (i)  $E$  contains all simple functions; (ii) if  $x \in E$  and  $y$  is a measurable function such that  $|y|$  is equi-distributed with  $|x|$ , then  $y \in E$  and  $\|x\|_E = \|y\|_E$ . For convenience we shall always assume  $E$  additionally satisfies

$$(1.1) \quad 0 \leq x_n \uparrow x, \quad x_n, x \in E \Rightarrow \|x_n\|_E \uparrow \|x\|_E.$$

For example,  $E$  satisfies (1.1) if it is  $\sigma$ -order continuous, i.e. for every sequence  $(x_n)_{n \geq 0}$  in  $E$ ,  $x_n \downarrow 0$  implies  $\|x_n\|_E \downarrow 0$ ; a fortiori  $E$  satisfies (1.1) if it fails to contain  $c_0$ .

A symmetric quasi Banach space  $E$  on  $[0,1]$  is said to have the Fatou property if for every net  $(x_i)_{i \in I}$  in  $E$  satisfying  $0 \leq x_i \uparrow$  and  $\sup_{i \in I} \|x_i\|_E < \infty$  the supremum  $x = \sup_{i \in I} x_i$  exists in  $E$  and  $\|x_i\|_E \uparrow \|x\|_E$ . We say that  $E$  has order continuous norm if for every net  $(x_i)_{i \in I}$  in  $E$  such that  $x_i \downarrow 0$  we have  $\|x_i\|_E \downarrow 0$ .

A symmetric Banach space  $E$  on  $[0,1]$  is called fully symmetric if, in addition, for  $x \in L_0([0,1])$  and  $y \in E$  with  $x \preceq y$  it follows that  $x \in E$  and  $\|x\|_E \leq \|y\|_E$ .

The Köthe dual of a symmetric Banach space  $E$  on  $[0,1]$  is the symmetric Banach space  $E^\times$  given by

$$\begin{aligned} E^\times &= \left\{ x \in L_0([0,1]) : \sup\left\{ \int_0^1 |x(t)y(t)| dt : \|x\|_E \leq 1 \right\} < \infty \right\}; \\ \|y\| &= \sup\left\{ \int_0^1 |x(t)y(t)| dt : \|x\|_E \leq 1 \right\}, \quad y \in E^\times. \end{aligned}$$

The space  $E^\times$  is fully symmetric and has the Fatou property. It is isometrically isomorphic to a closed subspace of  $E^*$  via the map

$$y \rightarrow L_y, \quad L_y(x) = \int_0^1 x(t)y(t) dt \quad (x \in E).$$

A symmetric Banach space  $E$  on  $[0,1]$  has the Fatou property if and only if  $E = E^{\times \times}$  isometrically. It has order continuous norm if and only if it is separable, which is also equivalent to the statement  $E^* = E^\times$ . Moreover, a symmetric Banach space which is separable or has the Fatou property is

automatically fully symmetric. Since we assumed symmetric Banach space  $E$  on  $[0, 1]$  is satisfying (1.1), so  $E$  embeds isometrically into its second Köthe dual  $E^{\times\times} = (E^\times)^\times$ .

For any  $0 < a < \infty$  we define the dilation operator  $D_a$  on  $L_0[0, 1]$  by

$$(D_a f)(s) = f(as)\chi_{[0,1]}(as) \quad (s \in [0, 1]).$$

If  $E$  is a symmetric Banach space on  $[0, 1]$ , then  $D_a$  is a bounded linear operator. Define the lower Boyd index  $p_E$  of  $E$  by

$$p_E = \sup\{p > 0 : \exists c > 0 \forall 0 < a \leq 1 \|D_a f\|_E \leq ca^{-\frac{1}{p}}\|f\|_E\}$$

and the upper Boyd index  $q_E$  of  $E$  by

$$q_E = \inf\{q > 0 : \exists c > 0 \forall a \geq 1 \|D_a f\|_E \leq ca^{-\frac{1}{q}}\|f\|_E\}.$$

It easily follows from the definitions that

$$1 \leq p_E \leq q_E \leq \infty, \quad p_{E^{(r)}} = rp_E, \quad q_{E^{(r)}} = rq_E.$$

If  $E$  is a symmetric Banach space on  $[0, 1]$ , then

$$(1.2) \quad \frac{1}{p_E} + \frac{1}{q_{E^\times}} = 1, \quad \frac{1}{p_{E^\times}} + \frac{1}{q_E} = 1.$$

We note that if  $E$  is  $p$ -convex then  $p_E \geq p$  and if  $E$  is  $q$ -concave then  $q_E \leq q$  (cf. [4, 5, 20, 23]).

Let  $E$  be a separable symmetric Banach space on  $[0, 1]$  with  $1 < p_E \leq q_E < \infty$ . Then, by Theorem 1.c.12 in [22] and Theorem 2.c.6 in [23],  $E$  is reflexive Banach space.

A quasi Banach space  $E$  is said to be a quasi Banach ideal space on  $[0, 1]$  if  $E$  is a linear subspace of  $L_0[0, 1]$  and satisfies the so-called ideal property, which means that if  $y \in E$ ,  $x \in L_0[0, 1]$  and  $|x(t)| \leq |y(t)|$  for  $\mu$ -almost all  $t \in \Omega$ , then  $x \in E$  and  $\|x\|_E \leq \|y\|_E$ .

Let  $E_i$  be a quasi Banach ideal space on  $[0, 1]$ ,  $i = 1, 2$ . We define the pointwise product space  $E_1 \odot E_2$  as

$$(1.3) \quad E_1 \odot E_2 = \{x : x = x_1 x_2, x_i \in E_i, i = 1, 2\}$$

with a functional  $\|x\|_{E_1 \odot E_2}$  defined by

$$\|x\|_{E_1 \odot E_2} = \inf\{\|x_1\|_{E_1}\|x_2\|_{E_2} : x = x_1 x_2, x_i \in E_i, i = 1, 2\}.$$

By Theorem 2 in [21], we know that if  $E_i$  is a quasi symmetric Banach space on  $[0, 1]$ ,  $i = 1, 2$ . Then  $E_1 \odot E_2$  is a quasi symmetric Banach space on  $[0, 1]$ .

We use the following result( see (iii) of Theorem 1 and Corollary 2 in [21]).

**Theorem 1.1.** *Let  $E, F$  be symmetric Banach spaces on  $[0, 1]$ .*

- (i) *If  $0 < p < \infty$ , then  $(E \odot F)^{(p)} = E^{(p)} \odot F^{(p)}$ .*
- (ii) *If  $1 < p < \infty$ , then  $(E^{(p)})^\times = (E^\times)^{(p)} \odot L_{p'}[0, 1]$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ .*

We also use the following Lozanovskii factorization theorem (Theorem 6 in [24]).

**Theorem 1.2.** *Let  $E$  be a symmetric Banach space on  $[0, 1]$ , then*

$$L_1[0, 1] = E \odot E^\times.$$

We use standard notation and notions from theory of noncommutative  $L^p$ -spaces. Our main references are [26] and [16] (see also [26] for more historical references). Throughout this paper, we denote by  $\mathcal{M}$  a finite von Neumann algebra on the Hilbert space  $\mathcal{H}$  with a faithful normal normalized finite trace  $\tau$ . The closed densely defined linear operator  $x$  in  $\mathcal{H}$  with domain  $D(x)$  is said to be affiliated with  $\mathcal{M}$  if and only if  $u^*xu = x$  for all unitary operators  $u$  which belong to the commutant  $\mathcal{M}'$  of  $\mathcal{M}$ . If  $x$  is affiliated with  $\mathcal{M}$ , then  $x$  is said to be  $\tau$ -measurable if for every  $\varepsilon > 0$  there exists a projection  $e \in \mathcal{M}$  such that  $e(\mathcal{H}) \subseteq D(x)$  and  $\tau(e^\perp) < \varepsilon$  (where for any projection  $e$  we let  $e^\perp = 1 - e$ ). The set of all  $\tau$ -measurable operators will be denoted by  $L_0(\mathcal{M})$ . The set  $L_0(\mathcal{M})$  is a  $*$ -algebra with sum and product being the respective closure of the algebraic sum and product.

Let  $x$  be a positive measurable operator and  $x = \int_0^\infty s de_s(x)$  its spectral decomposition. Composed with the spectral measure  $(e_s(x))$ , the trace  $\tau$  induces a positive measure  $d\tau(e_s(x))$  on  $\mathbb{R}^+$ . Thus we are reduced to the commutative case by regarding  $x$  as a function. In view of the preceding discussion

on functions, this justifies the following definition. Recall that  $e_s^\perp(x) = 1_{(s,\infty)}(x)$  is the spectral projection of  $x$  corresponding to the interval  $(s, \infty)$ .

Let  $x \in L_0(\mathcal{M})$ . Define

$$\lambda_s(x) = \tau(e_s^\perp(|x|)), \quad s > 0 \quad \text{and} \quad \mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leq t\}, \quad t > 0.$$

The function  $s \mapsto \lambda_s(x)$  is called the distribution function of  $x$  and the  $\mu_t(x)$  the generalized singular numbers of  $x$ . For more details on generalized singular value function of measurable operators we refer to [16].

Let  $E$  be a symmetric Banach space on  $[0, 1]$ . We define

$$L_E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \mu.(x) \in E\};$$

$$\|x\|_{L_E(\mathcal{M})} = \|\mu.(x)\|_E, \quad x \in L_E(\mathcal{M}).$$

Then  $(L_E(\mathcal{M}), \|\cdot\|_{L_E(\mathcal{M})})$  is a Banach space (cf. [13, 28]).

We need the following result (Theorem 5.6 in [15], p. 745).

**Theorem 1.3.** *Let  $E$  be a separable symmetric Banach space on  $[0, 1]$ , then  $L_E(\mathcal{M})^* = L_{E^\times}(\mathcal{M})$  isometrically, with associated duality bracket given by*

$$\langle x, y \rangle = \tau(xy) \quad (x \in L_E(\mathcal{M}), y \in L_{E^\times}(\mathcal{M})).$$

In what follows, unless otherwise specified, we always denote by  $E$  a symmetric Banach space on  $[0, 1]$ .

Let  $a = (a_n)_{n \geq 0}$  be a finite sequence in  $L_E(\mathcal{M})$ , define

$$\|a\|_{L_E(\mathcal{M}, \ell_c^2)} = \|(\sum_{n \geq 0} |a_n|^2)^{1/2}\|_{L_E(\mathcal{M})},$$

$$\|a\|_{L_E(\mathcal{M}, \ell_r^2)} = \|(\sum_{n \geq 0} |a_n^*|^2)^{1/2}\|_{L_E(\mathcal{M})}.$$

This gives two norms on the family of all finite sequences in  $L_E(\mathcal{M})$ . To see this, denoting by  $\mathcal{B}(\ell_2)$  the algebra of all bounded operators on  $\ell_2$  with its usual trace  $tr$ , let us consider the von Neumann algebra tensor product  $\mathcal{M} \otimes \mathcal{B}(\ell_2)$  with the product trace  $\tau \otimes tr$ .  $\tau \otimes tr$  is a semi-finite normal faithful trace, the associated noncommutative  $L_E$  space is denoted by  $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$ . Now, any finite sequence  $a = (a_n)_{n \geq 1}$  in  $L_E(\mathcal{M})$  can be regarded as an element in  $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$  via the following map

$$a \mapsto T(a) = \begin{pmatrix} a_0 & 0 & \dots \\ a_1 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

that is, the matrix of  $T(a)$  has all vanishing entries except those in the first column which are the  $a_n$ 's. Such a matrix is called a column matrix, and the closure in  $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$  of all column matrices is called the column subspace of  $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$ . Then

$$\|a\|_{L_E(\mathcal{M}, \ell_c^2)} = \|T(a)\|_{L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))} = \|T(a)\|_{L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))}.$$

Therefore  $\|\cdot\|_{L_E(\mathcal{M}, \ell_c^2)}$  defines a norm on the family of all finite sequences of  $L_E(\mathcal{M})$ . The corresponding completion is a Banach space, denoted by  $L_E(\mathcal{M}, \ell_c^2)$ . It is clear that if  $E$  has the Fatou property, then a sequence  $a = (a_n)_{n \geq 1}$  in  $L_E(\mathcal{M})$  belongs to  $L_E(\mathcal{M}, \ell_c^2)$  iff

$$\sup_{n \geq 1} \|(\sum_{k=1}^n |a_k|^2)^{1/2}\|_{L_E(\mathcal{M})} < \infty.$$

If this is the case,  $(\sum_{k=1}^\infty |a_k|^2)^{1/2}$  can be appropriately defined as an element of  $L_E(\mathcal{M})$ . Similarly, we may show that  $\|\cdot\|_{L_E(\mathcal{M}, \ell_r^2)}$  is a norm on the family of all finite sequence in  $L_E(\mathcal{M})$ . As above, it defines a Banach space  $L_E(\mathcal{M}, \ell_r^2)$ , which now is isometric to the row subspace of  $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$  consisting of matrices whose nonzero entries lie only in the first row. Observe that the column and row subspaces of  $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$  are 1-complemented subspace (by the definition of  $E$  and Theorem 3.4 in [14]).

We also need  $L_E^d(\mathcal{M})$ , the space of all sequences  $a = (a_n)_{n \geq 1}$  in  $L_E(\mathcal{M})$  such that

$$\|a\|_{L_E^d(\mathcal{M})} = \|\text{diag}(a_n)\|_{L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))} < \infty,$$

where  $\text{diag}(a_n)$  is the matrix with the  $a_n$  on the diagonal and zeroes elsewhere.

**1.1. Noncommutative martingales.** Let  $\mathcal{M}$  be a finite von Neumann algebra with a normalized normal faithful trace  $\tau$ . Let  $(\mathcal{M}_n)_{n \geq 1}$  be an increasing sequence of von Neumann subalgebras of  $\mathcal{M}$  such that  $\cup_{n \geq 1} \mathcal{M}_n$  generates  $\mathcal{M}$  (in the  $w^*$ -topology).  $(\mathcal{M}_n)_{n \geq 1}$  is called a filtration of  $\mathcal{M}$ . The restriction of  $\tau$  to  $\mathcal{M}_n$  is still denoted by  $\tau$ . Let  $\mathcal{E}_n = \mathcal{E}(\cdot | \mathcal{M}_n)$  be the conditional expectation of  $\mathcal{M}$  with respect to  $\mathcal{M}_n$ . Then  $\mathcal{E}_n$  is a norm 1 projection of  $L_E(\mathcal{M})$  onto  $L_E(\mathcal{M}_n)$  (see Theorem 3.4 in [14]) and  $\mathcal{E}_n(x) \geq 0$  whenever  $x \geq 0$ .

A noncommutative  $L_E$ -martingale with respect to  $(\mathcal{M}_n)_{n \geq 1}$  is a sequence  $x = (x_n)_{n \geq 1}$  such that  $x_n \in L_E(\mathcal{M}_n)$  and

$$\mathcal{E}_n(x_{n+1}) = x_n$$

for any  $n \geq 1$ . Let  $\|x\|_{L_E(\mathcal{M})} = \sup_{n \geq 1} \|x_n\|_{L_E(\mathcal{M})}$ . If  $\|x\|_{L_E(\mathcal{M})} < \infty$ , then  $x$  is said to be a bounded  $L_E$ -martingale.

Let  $x$  be a noncommutative martingale. The martingale difference sequence of  $x$ , denoted by  $dx = (dx_n)_{n \geq 1}$ , is defined as

$$dx_1 = x_1, \quad dx_n = x_n - x_{n-1}, \quad n \geq 2.$$

Set

$$S_n^c(x) = \left( \sum_{k=1}^n |dx_k|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad S_n^r(x) = \left( \sum_{k=1}^n |dx_k^*|^2 \right)^{\frac{1}{2}}.$$

By the preceding discussion, if  $E$  has the Fatou property, then  $dx$  belongs to  $L_E(\mathcal{M}, \ell_c^2)$  (resp.  $L_E(\mathcal{M}, \ell_r^2)$ ) if and only if  $(S_n^c(x))_{n \geq 1}$  (resp.  $(S_n^r(x))_{n \geq 1}$ ) is a bounded sequence in  $L_E(\mathcal{M})$ ; in this case,

$$S^c(x) = \left( \sum_{k=1}^{\infty} |dx_k|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad S^r(x) = \left( \sum_{k=1}^{\infty} |dx_k^*|^2 \right)^{\frac{1}{2}}$$

are elements in  $L_E(\mathcal{M})$ . These are noncommutative analogues of the usual square functions in the commutative martingale theory. It should be pointed out that the two sequences  $S_n^c(x)$  and  $S_n^r(x)$  may not be bounded in  $L_E(\mathcal{M})$  at the same time.

We define  $H_E^c(\mathcal{M})$  (resp.  $H_E^r(\mathcal{M})$ ) to be the space of all  $L_E$ -martingales such that  $dx \in L_E(\mathcal{M}, \ell_c^2)$  (resp.  $dx \in L_E(\mathcal{M}, \ell_r^2)$ ), equipped with the norm

$$\|x\|_{H_E^c(\mathcal{M})} = \|dx\|_{L_E(\mathcal{M}, \ell_c^2)} \quad (\text{resp. } \|x\|_{H_E^r(\mathcal{M})} = \|dx\|_{L_E(\mathcal{M}, \ell_r^2)}).$$

$H_E^c(\mathcal{M})$  and  $H_E^r(\mathcal{M})$  are Banach spaces. Note that if  $x \in H_E^c(\mathcal{M})$ ,

$$\|x\|_{H_E^c(\mathcal{M})} = \sup_{n \geq 0} \|S_n^c(x)\|_{L_E(\mathcal{M})} = \|S^c(x)\|_{L_E(\mathcal{M})}.$$

Similar equalities hold for  $H_E^r(\mathcal{M})$ .

We now consider the conditioned versions of square functions and Hardy spaces developed in [18]. For a finite noncommutative  $L_E$ -martingale  $x = (x_n)_{n \geq 1}$  define (with  $\mathcal{E}_0 = \mathcal{E}_1$ )

$$\|x\|_{h_E^c(\mathcal{M})} = \left\| \left( \sum_{k \geq 1} \mathcal{E}_{k-1}(|dx_k|^2) \right)^{\frac{1}{2}} \right\|_{L_E(\mathcal{M})}$$

and

$$\|x\|_{h_E^r(\mathcal{M})} = \left\| \left( \sum_{k \geq 1} \mathcal{E}_{k-1}(|dx_k^*|^2) \right)^{\frac{1}{2}} \right\|_{L_E(\mathcal{M})}.$$

Let  $h_E^c(\mathcal{M})$  and  $h_E^r(\mathcal{M})$  be the corresponding completions. Then  $h_E^c(\mathcal{M})$  and  $h_E^r(\mathcal{M})$  are Banach spaces. We define the column and row conditioned square functions as follows. For any finite martingale  $x = (x_n)_{n \geq 1}$  in  $L_E(\mathcal{M})$ , we set

$$s^c(x) = \left( \sum_{k \geq 1} \mathcal{E}_{k-1}(|dx_k|^2) \right)^{\frac{1}{2}} \quad \text{and} \quad s^r(x) = \left( \sum_{k \geq 1} \mathcal{E}_{k-1}(|dx_k^*|^2) \right)^{\frac{1}{2}}.$$

Then

$$\|x\|_{h_E^c(\mathcal{M})} = \|s^c(x)\|_{L_E(\mathcal{M})} \quad \text{and} \quad \|x\|_{h_E^r(\mathcal{M})} = \|s^r(x)\|_{L_E(\mathcal{M})}.$$

Let  $x = (x_n)_{n \geq 0}$  be a finite  $L_E$ -martingale, we set

$$s^d(x) = \text{diag}(|dx_n|)$$

We note that

$$\|s^d(x)\|_{L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))} = \|dx_n\|_{L_E^d(\mathcal{M})}$$

Let  $h_E^d(\mathcal{M})$  be the subspace of  $L_E^d(\mathcal{M})$  consisting of all martingale difference sequences.

**1.2. The space  $L_E(\mathcal{M}; \ell^\infty)$ .** Recall that  $L_E(\mathcal{M}; \ell^\infty)$  is defined as the space of all sequences  $(x_n)_{n \geq 1}$  in  $L_E(\mathcal{M})$  for which there exist  $a, b \in L_{E(2)}(\mathcal{M})$  and a bounded sequence  $(y_n)_{n \geq 1}$  in  $\mathcal{M}$  such that  $x_n = ay_nb$  for all  $n \geq 1$ . For such a sequence, set

$$(1.4) \quad \|(x_n)_{n \geq 1}\|_{L_E(\mathcal{M}, \ell^\infty)} := \inf \left\{ \|a\|_{L_{E(2)}(\mathcal{M})} \sup_n \|y_n\|_\infty \|b\|_{L_{E(2)}(\mathcal{M})} \right\},$$

where the infimum runs over all possible factorizations of  $(x_n)_{n \geq 1}$  as above. This is a norm and  $L_E(\mathcal{M}; \ell^\infty)$  is a Banach space (see [10]). As in [19], we usually write

$$\left\| \sup_n^+ x_n \right\|_E = \|(x_n)_{n \geq 1}\|_{L_E(\mathcal{M}, \ell^\infty)}.$$

We warn the reader that this suggestive notation should be treated with care. It is used for possibly nonpositive operators and

$$\left\| \sup_n^+ x_n \right\|_E \neq \left\| \sup_n^+ |x_n| \right\|_E$$

in general. However it has an intuitive description in the positive case: A positive sequence  $(x_n)_{n \geq 1}$  of  $L_E(\mathcal{M})$  belongs to  $L_E(\mathcal{M}; \ell^\infty)$  if and only if there exists a positive  $a \in L_E(\mathcal{M})$  such that  $x_n \leq a$  for any  $n \geq 1$  and in this case,

$$(1.5) \quad \left\| \sup_n^+ x_n \right\|_E \leq \inf \|a\|_{L_E(\mathcal{M})} \leq 2 \left\| \sup_n^+ x_n \right\|_E,$$

where the infimum runs over all possible positives  $a \in L_E(\mathcal{M})$  as above. Indeed, let  $(x_n)_{n \geq 1} \in L_E(\mathcal{M}; \ell^\infty)$ . Then for  $\varepsilon > 0$ , there exist  $a, b \in L_{E(2)}(\mathcal{M})^+$  and a bounded sequence  $(y_n)_{n \geq 1}$  in  $\mathcal{M}$  such that  $x_n = ay_nb$  for all  $n \geq 1$ ,  $\|a\|_{L_{E(2)}(\mathcal{M})} = \|b\|_{L_{E(2)}(\mathcal{M})}$ ,  $\sup_n \|y_n\|_\infty = 1$  and  $\|a\|_{L_{E(2)}(\mathcal{M})} \|b\|_{L_{E(2)}(\mathcal{M})} < \left\| \sup_n^+ x_n \right\|_E + \varepsilon$ . Define the operator  $c = (a^2 + b^2) \in L_E(\mathcal{M})^+$ . Then there are contractions  $u, v \in M$  such that  $a = c^{\frac{1}{2}}u$ ,  $b = vc^{\frac{1}{2}}$  (see Remark 2.3 in [8]). Hence,  $x_n = c^{\frac{1}{2}}\text{Re}(uy_nv)c^{\frac{1}{2}}$ , for all  $n \geq 1$ . It follows that  $x_n \leq c$  for all  $n \geq 1$  and  $\|c\|_{L_E(\mathcal{M})} < 2 \left\| \sup_n^+ x_n \right\|_E + 2\varepsilon$ . So, the second inequality of (1.5) holds. Conversely, if  $x_n \leq a$  for some  $a \in L_E^+(\mathcal{M})$ , then  $x_n^{\frac{1}{2}} = u_n a^{\frac{1}{2}}$  for a contraction  $u_n \in \mathcal{M}$ , and so  $x_n = a^{\frac{1}{2}}u_n^*u_n a^{\frac{1}{2}}$ . Thus  $\left\| \sup_n^+ x_n \right\|_E \leq \|a\|_{L_E(\mathcal{M})}$ . Hence, the first inequality of (1.5) holds.

We define  $L_E(\mathcal{M}; \ell^1)$  to be the space of all sequences  $x = (x_n)$  in  $L_E(\mathcal{M})$  which can be decomposed as

$$x_n = \sum_{k=1}^{\infty} u_{kn} v_{nk} \quad (n \geq 1)$$

for two families  $(u_{kn})_{k,n \geq 1}$  and  $(v_{nk})_{n,k \geq 1}$  in  $L_{E(2)}(\mathcal{M})$  satisfying

$$\sum_{k,n=1}^{\infty} u_{kn} u_{kn}^* \in L_E(\mathcal{M}) \quad \text{and} \quad \sum_{n,k=1}^{\infty} v_{nk}^* v_{nk} \in L_E(\mathcal{M}),$$

where the series converge in norm.  $L_E(\mathcal{M}; \ell^1)$  is a Banach space when equipped with the norm

$$\|x\|_{L_E(\mathcal{M}; \ell^1)} = \inf \left\{ \left\| \sum_{k,n=1}^{\infty} u_{kn} u_{kn}^* \right\|_{L_E(\mathcal{M})}^{1/2} \left\| \sum_{n,k=1}^{\infty} v_{nk}^* v_{nk} \right\|_{L_E(\mathcal{M})}^{1/2} \right\},$$

where the infimum runs over all decompositions of  $(x_n)$  as above. We will use the following fact (see [10]). Let  $x = (x_k)_{k \geq 1} \in L_E(\mathcal{M}; \ell^1)$  for which  $x_k \geq 0$  for all  $k$ . Then

$$(1.6) \quad \|x\|_{L_E(\mathcal{M}; \ell^1)} = \left\| \sum_{k \geq 1} x_k \right\|_{L_E(\mathcal{M})}.$$

We need the following result (Theorem 5.3 and in Remark 5.4 in [10]).

**Theorem 1.4.** *Let  $E$  be a separable symmetric Banach space on  $[0, 1]$ .*

(i) *If positive sequence  $(x_n)_{n \geq 1}$  of  $L_{E^\times}(\mathcal{M})$  belongs to  $L_{E^\times}(\mathcal{M}; \ell^\infty)$ , then*

$$(1.7) \quad \left\| \sup_n^+ x_n \right\|_{E^\times} = \sup \left\{ \sum_{k \geq 1} \tau(x_k y_k) : y_k \in L_E(\mathcal{M})^+, \left\| \sum_{k \geq 1} y_k \right\|_{L_E(\mathcal{M})} \leq 1 \right\}.$$

(ii)  *$L_E(\mathcal{M}; \ell^1)^* = L_{E^\times}(\mathcal{M}; \ell^\infty)$  isometrically, with respect to the duality bracket*

$$\langle x, y \rangle = \sum_{k \geq 1} \tau(x_k y_k),$$

where  $x \in L_E(\mathcal{M}; \ell^1)$  and  $y \in L_{E^\times}(\mathcal{M}; \ell^\infty)$ .

## 2. NORM VERSION OF NONCOMMUTATIVE BURKHOLDER-ROSENTHAL INEQUALITIES

**Lemma 2.1.** *Let  $E$  be a separable symmetric Banach space on  $[0, 1]$  with  $1 < p_E \leq q_E < 2$ . Then  $E = F \odot E^\times$ , where  $F = (E^{\times(\frac{1}{2})})^\times$  is separable.*

**Proof.** Since  $E^* = E^\times$ , by (1.2), it follows that  $2 < p_{E^\times} \leq q_{E^\times} < \infty$ . Hence  $1 < p_{E^{\times(\frac{1}{2})}} \leq q_{E^{\times(\frac{1}{2})}} < \infty$ . From the proof of Lemma 3.6 in [12], we know that  $E^{\times(\frac{1}{2})}$  is fully symmetric up to a constant. On the other hand, since  $E^\times$  has the Fatou property,  $E^{\times(\frac{1}{2})}$  has the Fatou property. Then, by Theorem 3.2 of [12],  $E^{\times(\frac{1}{2})}$  is an interpolation space for the couple  $(L_1[0, 1], L_\infty[0, 1])$ . By Theorem 3.4 in [12], there is a Banach function space  $G$  on  $(0, \infty)$  such that  $f \in E^{\times(\frac{1}{2})}$  if and only if  $t \mapsto K(t, f; L_1, L_\infty) \in G$  and there exist constants  $c, C > 0$  such that

$$c \|t \mapsto K(t, f; L_1, L_\infty)\|_G \leq \|f\|_{E^{\times(\frac{1}{2})}} \leq C \|t \mapsto K(t, f; L_1, L_\infty)\|_G,$$

where

$$K(t, f; L_1, L_\infty) = \inf_{f=f_0+f_1} \{ \|f_0\|_{L_1[0,1]} + t \|f_1\|_{L_\infty[0,1]} \} \quad (t > 0).$$

Set

$$\|f\|'_{E^{\times(\frac{1}{2})}} = \|t \mapsto K(t, f; L_1, L_\infty)\|_G, \quad \forall f \in E^{\times(\frac{1}{2})}.$$

Then  $(E^{\times(\frac{1}{2})}, \|\cdot\|'_{E^{\times(\frac{1}{2})}})$  is a Banach space and an interpolation space for the couple  $(L_1[0, 1], L_\infty[0, 1])$ .

Hence,  $E^{\times(\frac{1}{2})}$  can be renormed, with an equivalent norm,  $E^{\times(\frac{1}{2})}$  becomes an exact interpolation space for the couple  $(L_1[0, 1], L_\infty[0, 1])$  (see [14], p.944). So, with this equivalent norm,  $E^{\times(\frac{1}{2})}$  becomes a fully symmetric Banach space.

Let  $F = (E^{\times(\frac{1}{2})})^\times$ . Since  $E^{\times(\frac{1}{2})}$  has the Fatou property,  $F^\times = E^{\times(\frac{1}{2})}$ . Notice that  $E$  is a separable symmetric Banach space on  $[0, 1]$  with  $1 < p_E \leq q_E < 2$ , so  $E$  is reflexive. Hence  $E^* = E^\times$  is separable. Therefore,  $E^{\times(\frac{1}{2})}$  is separable. It follows that  $E^{\times(\frac{1}{2})}$  is reflexive and  $F$  is separable.

By (ii) of Theorem 1.1, we have

$$E = (E^\times)^\times = ([E^{\times(\frac{1}{2})}]^{(2)})^\times = ([E^{\times(\frac{1}{2})}]^\times)^{(2)} \odot L^2[0, 1] = F^{(2)} \odot L^2[0, 1].$$

Using Theorem 1.2 and (i) of Theorem 1.1 we obtain that

$$E = F^{(2)} \odot L^2[0, 1] = F^{(2)} \odot [F \odot E^{\times(\frac{1}{2})}]^{(2)} = F^{(2)} \odot [F^{(2)} \odot E^\times] = F \odot E^\times.$$

□

**Lemma 2.2.** *Let  $E, E_1, E_2$  be separable symmetric Banach spaces on  $[0, 1]$  such that  $E = E_1 \odot E_2$ . If  $x \in L_E(\mathcal{M})^+$ , then for  $\varepsilon > 0$ , there exist  $a \in L_{E_1}^+(\mathcal{M})$  and  $b \in L_{E_2}^+(\mathcal{M})$  such that  $x = ab$ ,  $\|a\|_{L_{E_1}(\mathcal{M})}\|b\|_{L_{E_2}(\mathcal{M})} < \|x\|_{L_E(\mathcal{M})} + \varepsilon$  and  $a$  is invertible with bounded inverse.*

**Proof.** Let  $\mathcal{N}$  be the commutative von Neumann subalgebra of  $\mathcal{M}$  generated by the spectral projection of  $x$ . Then  $\mathcal{N}$  is isometric isomorphic to  $L_\infty(\Omega, \Sigma, \mu)$  where  $(\Omega, \Sigma, \mu)$  a finite measure space. Hence,  $x \in L_E(\mathcal{N}) = L_E(\Omega, \mu)$ . Since  $E = E_1 \odot E_2$ , for every  $\varepsilon > 0$ , there are  $x_1 \in L_{E_1}(\Omega, \mu)^+ = L_{E_1}(\mathcal{N})^+$  and  $x_2 \in L_{E_2}(\Omega, \mu)^+ = L_{E_2}(\mathcal{N})^+$  such that  $x = x_1 x_2$  and  $\|x\|_{L_E(\mathcal{N})} + \frac{\varepsilon}{2} > \|x_1\|_{L_{E_1}(\mathcal{N})}\|x_2\|_{L_{E_2}(\mathcal{N})}$ . Let  $\delta > 0$ . Set  $a = x_1 + \delta$  and  $b = x_1(x_1 + \delta)^{-1}x_2$ . Then  $a \in L_{E_1}(\mathcal{N}) \subset L_{E_1}(\mathcal{M})$ ,  $b \in L_{E_2}(\mathcal{N}) \subset L_{E_2}(\mathcal{M})$ ,  $x = ab$  and  $a$  is invertible with bounded inverse. For enough small  $\delta$  we have that  $\|x\|_{L_E(\mathcal{M})} + \varepsilon > \|a\|_{L_{E_1}(\mathcal{M})}\|b\|_{L_{E_2}(\mathcal{M})}$ . Hence we obtain the desired result.  $\square$

**Theorem 2.3.** *Let  $E$  be a separable symmetric Banach space on  $[0, 1]$  with  $1 < p_E \leq q_E < 2$ . Then we have*

- (i)  $(h_E^c(\mathcal{M}))^* = h_{E^\times}^c(\mathcal{M})$  with equivalent norms.
- (ii)  $(H_E^c(\mathcal{M}))^* = H_{E^\times}^c(\mathcal{M})$  with equivalent norms.

Similarly,  $(h_E^r(\mathcal{M}))^* = h_{E^\times}^r(\mathcal{M})$  and  $(H_E^r(\mathcal{M}))^* = H_{E^\times}^r(\mathcal{M})$  with equivalent norms.

**Proof.** (i)  $1^\circ$  From  $1 < p_{E^\times} \leq q_{E^\times} < 2$ , we obtain that  $L_2(\mathcal{M}) \subset L_E(\mathcal{M})$  with continuous inclusions, i.e. there exists a constant  $C > 0$  such that  $\|x\|_E \leq C\|x\|_{L_2(\mathcal{M})}$  for all  $x \in L_2(\mathcal{M})$ . We identify an element  $x \in L_2(\mathcal{M})$  with the martingale  $(\mathcal{E}_n(x))_{n \geq 1}$ . By the trace-preserving property of conditional expectations and the orthogonality in  $L_2(\mathcal{M})$  of martingale difference sequences, we get

$$\begin{aligned} \|x\|_{h_E^c(\mathcal{M})} &= \|(\sum_{k \geq 1} \mathcal{E}_{k-1}(|dx_k|^2))^{\frac{1}{2}}\|_{L_E(\mathcal{M})} \\ &\leq C\|(\sum_{k \geq 1} \mathcal{E}_{k-1}(|dx_k|^2))^{\frac{1}{2}}\|_{L_2(\mathcal{M})} \\ &= C\|x\|_{L_2(\mathcal{M})}, \end{aligned}$$

i.e. this martingale is in  $h_E^c(\mathcal{M})$ .

Let  $y \in h_{E^\times}^c(\mathcal{M})$ . Since  $2 < p_{E^\times} \leq q_{E^\times} < \infty$ , it follows that  $E^\times \subset L_2([0, 1])$  with continuous inclusions. Hence,  $\|y\|_{L_2(\mathcal{M})} \leq C_1\|y\|_{h_{E^\times}^c(\mathcal{M})} < \infty$ , i.e.  $y$  is an  $L_2$ -martingale. Define  $\phi_y$  by  $\phi_y(x) = \tau(y^*x)$ ,  $\forall x \in L_2(\mathcal{M})$ . We must show that  $\phi_y$  induces a continuous linear functional on  $h_E^c(\mathcal{M})$ .

By Lemma 2.1, there is a separable symmetric Banach space  $F$  on  $[0, 1]$  such that  $F^\times = E^{\times(\frac{1}{2})}$  and  $E = F \odot E^\times$ . Using Lemma 2.2, we obtain that for  $\varepsilon > 0$ , there exist  $a \in L_F^+(\mathcal{M})$  and  $b \in L_{E^\times}^+(\mathcal{M})$  such that  $s^c(x) = ab$ ,  $\|a\|_{L_F(\mathcal{M})}\|b\|_{L_{E^\times}(\mathcal{M})} < \|s^c(x)\|_{L_E(\mathcal{M})} + \varepsilon$  and  $a$  is invertible with bounded inverse. Then, by the Cauchy-Schwarz inequality and the tracial property of  $\tau$ , we have

$$\begin{aligned} |\phi_y(x)| &= |\sum_{n \geq 1} \tau(dy_n^* dx_n)| = |\sum_{n \geq 1} \tau(\mathcal{E}_{n-1}(a)^{\frac{1}{2}} dy_n^* dx_n \mathcal{E}_{n-1}(a)^{-\frac{1}{2}})| \\ &\leq \left[ \sum_{n \geq 1} \tau(\mathcal{E}_{n-1}(a)^{\frac{1}{2}} |dy_n|^2 \mathcal{E}_{n-1}(a)^{\frac{1}{2}}) \right]^{\frac{1}{2}} \left[ \sum_{n \geq 1} \tau(\mathcal{E}_{n-1}(a)^{-\frac{1}{2}} |dx_n|^2 \mathcal{E}_{n-1}(a)^{-\frac{1}{2}}) \right]^{\frac{1}{2}} \\ &= \left[ \sum_{n \geq 1} \tau(a \mathcal{E}_{n-1}(|dy_n|^2)) \right]^{\frac{1}{2}} \left[ \sum_{n \geq 1} \tau(\mathcal{E}_{n-1}(a)^{-1} |dx_n|^2) \right]^{\frac{1}{2}} \\ &= \text{I} \cdot \text{II}. \end{aligned}$$

By Theorem 1.3, we have

$$\begin{aligned} \text{I}^2 &= \sum_{n \geq 1} \tau(a \mathcal{E}_{n-1}(|dy_n|^2)) = \tau(a \sum_{n \geq 1} \mathcal{E}_{n-1}(|dy_n|^2)) \\ &\leq \|a\|_{L_F(\mathcal{M})} \|\sum_{n \geq 1} \mathcal{E}_{n-1}(|dy_n|^2)\|_{L_{E^{\times(\frac{1}{2})}}(\mathcal{M})} \\ &= \|a\|_{L_F(\mathcal{M})} \|(\sum_{n \geq 1} \mathcal{E}_{n-1}(|dy_n|^2))^{\frac{1}{2}}\|_{L_{E^\times}(\mathcal{M})}^2 \\ &= \|a\|_{L_F(\mathcal{M})} \|y\|_{h_{E^\times}^c(\mathcal{M})}^2. \end{aligned}$$

To estimate II, we set

$$s_{c,n}(x) = \left( \sum_{k=1}^n \mathcal{E}_{k-1}(|dx_k|^2) \right)^{\frac{1}{2}}, \quad \forall n \geq 1$$

and  $s_{c,0}(x) = |dx_1|$ . Then

$$\sum_{n \geq 1} \mathcal{E}_{n-1}(|dx_n|^2) = \sum_{n \geq 0} (s_{c,n+1}(x)^2 - s_{c,n}(x)^2).$$



Applying Corollary 2.3 of [7] we obtain that  $\mathcal{E}_{n-1}(a^{-1}) \geq \mathcal{E}_{n-1}(a)^{-1}$ . Hence, by Theorem 1.3, we have

$$\begin{aligned} \Pi^2 &= \sum_{n \geq 1} \tau(\mathcal{E}_{n-1}(a)^{-1} |dx_n|^2) \leq \sum_{n \geq 1} \tau(\mathcal{E}_{n-1}(a^{-1}) |dx_n|^2) \\ &= \sum_{n \geq 1} \tau(a^{-1} \mathcal{E}_{n-1}(|dx_n|^2)) = \sum_{n \geq 0} \tau(a^{-1} (s_{c,n+1}(x)^2 - s_{c,n}(x)^2)) \\ &= \tau(a^{-1} s^c(x)^2) = \tau(bs^c(x)) \\ &\leq \|b\|_{L_{E^\times}(\mathcal{M})} \|s^c(x)\|_{L_E(\mathcal{M})} = \|b\|_{L_{E^\times}(\mathcal{M})} \|x\|_{h_E^c(\mathcal{M})}. \end{aligned}$$

Combining the precedent estimations, for any finite  $L_2$ -martingale  $x$ , we deduce that

$$|\phi_y(x)| \leq \|y\|_{h_{E^\times}^c(\mathcal{M})} \|x\|_{h_E^c(\mathcal{M})}$$

Thus  $\phi$  extends to an element of  $h_E^c(\mathcal{M})^*$  with norm at most  $\|y\|_{h_{E^\times}^c(\mathcal{M})}$ .

2° Let  $\phi \in h_E^c(\mathcal{M})^*$  of norm one. As  $L_2(\mathcal{M}) \subset h_E^c(\mathcal{M})$ , it follows that  $\phi$  induces a continuous functional  $\tilde{\phi}$  on  $L_2(\mathcal{M})$ . Consequently,  $\tilde{\phi}$  is given by an element  $y$  of  $L_2(\mathcal{M})$ ,

$$\tilde{\phi}(y) = \tau(y^*x), \quad \forall x \in L_2(\mathcal{M}).$$

As finite  $L_2$ -martingales are dense in  $h_E^c(\mathcal{M})$  and in  $L_2(\mathcal{M})$ , we deduce that  $L_2(\mathcal{M})$  is dense in  $h_E^c(\mathcal{M})$ . We have

$$(2.1) \quad \|\phi\|_{h_E^c(\mathcal{M})^*} = \sup_{x \in L_2(\mathcal{M}), \|x\|_{h_E^c(\mathcal{M})} \leq 1} |\tau(y^*x)| \leq 1.$$

We want to show that  $y \in h_{E^\times}^c(\mathcal{M})$  and  $\|y\|_{h_{E^\times}^c(\mathcal{M})} \leq C$ .

Set

$$z_n = \mathcal{E}_{n-1}(|dy_n|^2), \quad \forall n \geq 1.$$

Using Theorem 1.3 we obtain that

$$\begin{aligned} \|y\|_{h_{E^\times}^c(\mathcal{M})}^2 &= \|\sum_{n \geq 1} \mathcal{E}_{n-1}(|dy_n|^2)\|_{L_{E^\times(\frac{1}{2})}(\mathcal{M})} = \|\sum_{n \geq 1} z_n\|_{L_{E^\times(\frac{1}{2})}(\mathcal{M})} \\ &= \sup \left\{ \sum_{n \geq 1} \tau(z_n a) : a \in L_F^+(\mathcal{M}) \text{ and } \|a\|_{L_F(\mathcal{M})} \leq 1 \right\} \\ &= \sup \left\{ \sum_{n \geq 1} \tau(z_n \mathcal{E}_{n-1}(a)) : a \in L_F^+(\mathcal{M}) \text{ and } \|a\|_{L_F(\mathcal{M})} \leq 1 \right\}, \end{aligned}$$

where  $F$  is a separable symmetric Banach space on  $[0, 1]$  such that  $F^\times = E^{\times(\frac{1}{2})}$ .

Let  $a \in L_F^+(\mathcal{M})$  and  $\|a\|_F \leq 1$ . Let  $b$  be the martingale defined as follows:

$$db_n = dy_n \mathcal{E}_{n-1}(a), \quad \forall n \geq 1.$$

Using (2.1) we obtain that

$$\tau(y^*b) \leq \|b\|_{h_E^c(\mathcal{M})}.$$

We have that

$$\tau(y^*b) = \sum_{n \geq 1} \tau(|dy_n|^2 \mathcal{E}_{n-1}(a)) = \sum_{n \geq 1} \tau(\mathcal{E}_{n-1}(|dy_n|^2) \mathcal{E}_{n-1}(a)) = \sum_{n \geq 1} \tau(z_n \mathcal{E}_{n-1}(a)).$$

On the other hand, by the definition of  $b$ , we find

$$\begin{aligned} s^c(b)^2 &= \sum_{n \geq 1} \mathcal{E}_{n-1}(\mathcal{E}_{n-1}(a) |dy_n|^2 \mathcal{E}_{n-1}(a)) \\ &= \sum_{n \geq 1} \mathcal{E}_{n-1}(a) \mathcal{E}_{n-1}(|dy_n|^2) \mathcal{E}_{n-1}(a) \\ &= \sum_{n \geq 1} \mathcal{E}_{n-1}(a) z_n \mathcal{E}_{n-1}(a). \end{aligned}$$

Let  $c \in L_F^+(\mathcal{M})$  such that  $\mathcal{E}_{n-1}(a) \leq c$  for all  $n \geq 1$ . Then for each  $n$ , there exists a contraction  $u_n \in \mathcal{M}$  such that  $\mathcal{E}_{n-1}(a)^{\frac{1}{2}} = u_n c^{\frac{1}{2}}$ , and so

$$s^c(b)^2 = \sum_{n \geq 1} c^{\frac{1}{2}} u_n^* \mathcal{E}_{n-1}(a)^{\frac{1}{2}} z_n \mathcal{E}_{n-1}(a)^{\frac{1}{2}} u_n c^{\frac{1}{2}} = c^{\frac{1}{2}} \left( \sum_{n \geq 1} u_n^* \mathcal{E}_{n-1}(a)^{\frac{1}{2}} z_n \mathcal{E}_{n-1}(a)^{\frac{1}{2}} u_n \right) c^{\frac{1}{2}}.$$

Whence

$$\mu_t(s^c(b)^2) \leq \mu_{\frac{t}{3}}(c^{\frac{1}{2}}) \mu_{\frac{t}{3}} \left( \sum_{n \geq 1} u_n^* \mathcal{E}_{n-1}(a)^{\frac{1}{2}} z_n \mathcal{E}_{n-1}(a)^{\frac{1}{2}} u_n \right) \mu_{\frac{t}{3}}(c^{\frac{1}{2}}), \quad \forall t \in [0, t].$$

Since  $E^{(\frac{1}{2})} = F^{(2)} \odot L^1[0, 1] \odot F^{(2)}$ , we have that

$$\begin{aligned} \|b\|_{h_E^c(\mathcal{M})}^2 &= \|s^c(b)\|_{L_E(\mathcal{M})}^2 = \|s^c(b)^2\|_{L_{E^{(\frac{1}{2})}}(\mathcal{M})} = \|\mu_t(s^c(b)^2)\|_{E^{(\frac{1}{2})}} \\ &\leq \|\mu_{\frac{t}{3}}(c^{\frac{1}{2}})\mu_{\frac{t}{3}}(\sum_{n \geq 1} u_n^* \mathcal{E}_{n-1}(a)^{\frac{1}{2}} z_n \mathcal{E}_{n-1}(a)^{\frac{1}{2}} u_n) \mu_{\frac{t}{3}}(c^{\frac{1}{2}})\|_{E^{(\frac{1}{2})}} \\ &\leq \|\mu_{\frac{t}{3}}(c^{\frac{1}{2}})\|_{F^{(2)}} \|\mu_{\frac{t}{3}}(\sum_{n \geq 1} u_n^* \mathcal{E}_{n-1}(a)^{\frac{1}{2}} z_n \mathcal{E}_{n-1}(a)^{\frac{1}{2}} u_n)\|_1 \|\mu_{\frac{t}{3}}(c^{\frac{1}{2}})\|_{F^{(2)}} \\ &\leq K \|c\|_F^{\frac{1}{2}} \left( \sum_{n \geq 1} \tau(u_n^* \mathcal{E}_{n-1}(a)^{\frac{1}{2}} z_n \mathcal{E}_{n-1}(a)^{\frac{1}{2}} u_n) \right) \|c\|_F^{\frac{1}{2}} \\ &\leq K \|c\|_F \left( \sum_{n \geq 1} \tau(z_n \mathcal{E}_{n-1}(a)) \right), \end{aligned}$$

where  $K$  is a constant depending only on norms of the delation operator on the spaces  $F^{(2)}$ ,  $L^1[0, 1]$ . Using (1.5) and Corollary 5.4 of [11] we deduce that

$$\|b\|_{h_E^c(\mathcal{M})} \leq C \|a\|_F^{\frac{1}{2}} \left( \sum_{n \geq 1} \tau(z_n \mathcal{E}_{n-1}(a)) \right)^{\frac{1}{2}},$$

where  $C$  is a constant. Combining the preceding inequalities, we obtain that

$$\sum_{n \geq 1} \tau(z_n \mathcal{E}_{n-1}(a)) \leq C^2 \|a\|_F \leq C^2.$$

It follows that  $\|y\|_{h_E^c(\mathcal{M})} \leq C$ . Thus we have finished the proof of (i).

(ii) 1° Let  $y \in H_{E \times}^c(\mathcal{M})$  and define  $\phi_y$  by  $\phi_y(x) = \tau(y^*x)$ ,  $\forall x \in L_2(\mathcal{M})$ . We must show that  $\phi_y$  induces a continuous linear functional on  $H_E^c(\mathcal{M})$ . Let  $x$  be a finite  $L_2$ -martingale such that  $\|x\|_{H_E^c(\mathcal{M})} < \infty$ .

$$\begin{aligned} |\phi_y(x)| &= |\sum_{n \geq 1} \tau(dy_n^* dx_n)| = |\tau \otimes \text{tr} \left( \begin{pmatrix} dy_1 & 0 & \cdots \\ dy_2 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}^* \begin{pmatrix} dx_1 & 0 & \cdots \\ dx_2 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \right)| \\ &\leq \left\| \begin{pmatrix} dy_1 & 0 & \cdots \\ dy_2 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}^* \right\|_{L_{E \times}(\mathcal{M} \otimes \mathcal{B}(\ell_2))} \left\| \begin{pmatrix} dx_1 & 0 & \cdots \\ dx_2 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \right\|_{L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))} \\ &= \|y\|_{H_{E \times}^c(\mathcal{M})} \|x\|_{H_E^c(\mathcal{M})}. \end{aligned}$$

Thus  $\phi_y$  extends to an element of  $H_E^c(\mathcal{M})^*$  with norm at most  $\|y\|_{H_{E \times}^c(\mathcal{M})}$ .

2° Let  $\phi \in H_E^c(\mathcal{M})^*$  be such that  $\|\phi\|_{H_E^c(\mathcal{M})^*} \leq 1$ . There exists  $y \in L_2(\mathcal{M})$  such that

$$\phi(y) = \tau(y^*x), \quad \forall x \in L_2(\mathcal{M}).$$

By the density of  $L_2(\mathcal{M})$  in  $H_E^c(\mathcal{M})$ , we have

$$(2.2) \quad \|\phi\|_{H_E^c(\mathcal{M})^*} = \sup_{x \in L_2(\mathcal{M}), \|x\|_{H_E^c(\mathcal{M})} \leq 1} |\tau(y^*x)| \leq 1.$$

We use the same method in 2° of the proof of (i) to show that  $y \in H_{E \times}^c(\mathcal{M})$  and  $\|y\|_{H_{E \times}^c(\mathcal{M})} \leq C$ . By Theorem 1.3, we find that

$$\begin{aligned} \|y\|_{H_{E \times}^c(\mathcal{M})}^2 &= \|\sum_{n \geq 1} |dy_n|^2\|_{L_{E \times}(\frac{1}{2})}(\mathcal{M}) \\ &= \sup \left\{ \sum_{n \geq 1} \tau(|dy_n|^2 a) : a \in L_F^+(\mathcal{M}) \text{ and } \|a\|_{L_F(\mathcal{M})} \leq 1 \right\} \\ &= \sup \left\{ \sum_{n \geq 1} \tau(|dy_n|^2 \mathcal{E}_n(a)) : a \in L_F^+(\mathcal{M}) \text{ and } \|a\|_{L_F(\mathcal{M})} \leq 1 \right\}. \end{aligned}$$

Let  $a \in L_F^+(\mathcal{M})$  and  $\|a\|_F \leq 1$ . Set

$$db_n = dy_n \mathcal{E}_n(a) - \mathcal{E}_{n-1}(dy_n \mathcal{E}_n(a)), \quad \forall n \geq 1.$$

Then  $b$  is a martingale. By (2.2), it follows that

$$\tau(y^*b) \leq \|b\|_{H_E^c(\mathcal{M})}.$$

Since  $(dy_n)_{n \geq 1}$  is a martingale difference sequence, we obtain that

$$\begin{aligned} \tau(y^*b) &= \sum_{n \geq 1} \tau(|dy_n|^2 \mathcal{E}_n(a)) - \sum_{n \geq 1} \tau(dy_n^* \mathcal{E}_{n-1}(dy_n \mathcal{E}_n(a))) \\ &= \sum_{n \geq 1} \tau(|dy_n|^2 \mathcal{E}_n(a)) - \sum_{n \geq 1} \tau(\mathcal{E}_{n-1}(dy_n^*) dy_n \mathcal{E}_n(a)) \\ &= \sum_{n \geq 1} \tau(|dy_n|^2 \mathcal{E}_n(a)). \end{aligned}$$

Using the triangular inequality in  $L_E(\mathcal{M}, \ell_c^2)$ , we get

$$\begin{aligned} \|b\|_{H_E^c(\mathcal{M})} &= \|(db_n)_{n \geq 1}\|_{L_E(\mathcal{M}, \ell_c^2)} \\ &\leq \|(dy_n \mathcal{E}_n(a))_{n \geq 1}\|_{L_E(\mathcal{M}, \ell_c^2)} + \|(\mathcal{E}_{n-1}(dy_n \mathcal{E}_n(a)))_{n \geq 1}\|_{L_E(\mathcal{M}, \ell_c^2)}. \end{aligned}$$

On the other hand, by Lemma 2.2 of [1], there is a constant  $r_E$  such that

$$\|(\mathcal{E}_{n-1}(dy_n \mathcal{E}_n(a)))_{n \geq 1}\|_{L_E(\mathcal{M}, \ell_c^2)} \leq r_E \|(dy_n \mathcal{E}_n(a))_{n \geq 1}\|_{L_E(\mathcal{M}, \ell_c^2)}.$$

Hence,

$$\|b\|_{H_E^c(\mathcal{M})} \leq (1 + r_E) \|(dy_n \mathcal{E}_n(a))_{n \geq 1}\|_{L_E(\mathcal{M}, \ell_c^2)}.$$

Let  $c \in L_F^+(\mathcal{M})$  such that  $\mathcal{E}_n(a) \leq c$  for all  $n \geq 1$ . Then for each  $n$ , there exists a contraction  $u_n \in \mathcal{M}$  such that  $\mathcal{E}_n(a)^{\frac{1}{2}} = u_n c^{\frac{1}{2}}$ . As before, by  $E^{(\frac{1}{2})} = F^{(2)} \odot L^1[0, 1] \odot F^{(2)}$ , we have that

$$\begin{aligned} \|(dy_n \mathcal{E}_n(a))_{n \geq 1}\|_{L_E(\mathcal{M}, \ell_c^2)}^2 &= \|c^{\frac{1}{2}} \left( \sum_{n \geq 1} u_n^* \mathcal{E}_n(a)^{\frac{1}{2}} |dy_n|^2 \mathcal{E}_n(a)^{\frac{1}{2}} u_n \right) c^{\frac{1}{2}}\|_{L_{E^{(\frac{1}{2})}}(\mathcal{M})}^2 \\ &\leq K \|c\|_F^{\frac{1}{2}} \left( \sum_{n \geq 1} \tau(u_n^* \mathcal{E}_n(a)^{\frac{1}{2}} |dy_n|^2 \mathcal{E}_n(a)^{\frac{1}{2}} u_n) \right) \|c\|_F^{\frac{1}{2}} \\ &\leq K \|c\|_F \left( \sum_{n \geq 1} \tau(|dy_n|^2 \mathcal{E}_n(a)) \right). \end{aligned}$$

By (1.5) and Corollary 5.4 of [11] we deduce that

$$\|b\|_{H_E^c(\mathcal{M})} \leq C \|a\|_F^{\frac{1}{2}} \left( \sum_{n \geq 1} \tau(|dy_n|^2 \mathcal{E}_n(a)) \right)^{\frac{1}{2}},$$

where  $C$  is a constant. Combining the preceding inequalities, we obtain that

$$\tau \left( \sum_{n \geq 1} |dy_n|^2 a \right) = \sum_{n \geq 1} \tau(|dy_n|^2 \mathcal{E}_n(a)) \leq C^2 \|a\|_F \leq C^2.$$

It follows that  $\|y\|_{H_E^c(\mathcal{M})} \leq C$ . Thus we have finished the proof of (ii).

Passing to adjoint, we obtain the identities  $(h_E^r(\mathcal{M}))^* = h_{E^\times}^r(\mathcal{M})$  and  $(H_E^r(\mathcal{M}))^* = H_{E^\times}^r(\mathcal{M})$ .  $\square$

**Lemma 2.4.** *Let  $E$  be a separable symmetric Banach space on  $[0, 1]$ . Then we have  $(h_E^d(\mathcal{M}))^* = h_{E^\times}^d(\mathcal{M})$  with equivalent norms.*

**Proof.** Recall that  $h_E^d(\mathcal{M})$  consists of martingale difference sequences in  $L_E^d(\mathcal{M})$ . So  $h_E^d(\mathcal{M})$  is 2-complemented in  $L_E^d(\mathcal{M})$  via the projection

$$P : \begin{cases} L_E^d(\mathcal{M}) & \longrightarrow & h_E^d(\mathcal{M}) \\ (a_n)_{n \geq 1} & \longmapsto & (\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n))_{n \geq 1} \end{cases}$$

By Theorem 1.3, we obtain the desired result.  $\square$

**Proposition 2.5.** *Let  $E$  be a separable symmetric Banach space on  $[0, 1]$  with  $2 < p_E \leq q_E < \infty$ . Then we have*

- (i)  $H_E^c(\mathcal{M}) = h_E^c(\mathcal{M}) \cap h_E^d(\mathcal{M})$  with equivalent norms.
- (ii)  $H_E^r(\mathcal{M}) = H_E^r(\mathcal{M}) \cap h_E^d(\mathcal{M})$  with equivalent norms.

**Proof.** (i) Let  $y \in H_E^c(\mathcal{M})$ . From the proof of Theorem 6.2 in [10] we have that

$$\|y\|_{H_E^c(\mathcal{M})} \leq \|y\|_{h_E^c(\mathcal{M}) \cap h_E^d(\mathcal{M})}.$$

On the other hand, from the proof of Theorem 6.2 in [10], it follows that

$$\|dy_n\|_{L_E^d(\mathcal{M})} \leq C_1 \sum_{n \geq 1} r_n \otimes |dx_n|_{L_E(L^\infty \overline{\otimes} \mathcal{M})}.$$

Hence, by Theorem 4.7 in [12], we get  $\|dy_n\|_{L_E^d(\mathcal{M})} \leq C_2 \|y\|_{H_E^c(\mathcal{M})}$ . Since  $p_{E^{(\frac{1}{2})}} = \frac{1}{2} p_E > 1$ , by Corollary 4.13 in [10], we have  $\|y\|_{h_E^c(\mathcal{M})} \leq C_3 \|y\|_{H_E^c(\mathcal{M})}$ . Thus

$$\|y\|_{h_E^c(\mathcal{M}) \cap h_E^d(\mathcal{M})} \leq C' \|y\|_{H_E^c(\mathcal{M})}.$$

(ii) Passing to adjoint, we obtain the desired result.  $\square$

**Proposition 2.6.** *Let  $E$  be a separable symmetric Banach space on  $[0, 1]$  with  $1 < p_E \leq q_E < 2$ . Then we have*

- (i)  $H_E^c(\mathcal{M}) = h_E^c(\mathcal{M}) + h_E^d(\mathcal{M})$  with equivalent norms.

(ii)  $H_E^r(\mathcal{M}) = h_E^r(\mathcal{M}) + h_E^d(\mathcal{M})$  with equivalent norms.

**Proof.** (i) By Proposition 2.5, it follows that there exist constants  $C > 0$  such that

$$(2.3) \quad \max\{\|x\|_{h_{E^\times}^c(\mathcal{M})}, \|dy_n\|_{L_{E^\times}^d(\mathcal{M})}\} \leq C\|x\|_{H_{E^\times}^c(\mathcal{M})}, \quad \forall x \in H_{E^\times}^c(\mathcal{M}).$$

Let  $y \in h_E^c(\mathcal{M})$ . By (2.3) and Theorem 2.3, we deduce that

$$\|y\|_{H_E^c(\mathcal{M})} = \sup_{\|x\|_{H_{E^\times}^c(\mathcal{M})} \leq 1} |\tau(x^*y)| \leq \sup_{\|x\|_{h_{E^\times}^c(\mathcal{M})} \leq C} |\tau(x^*y)| = C\|y\|_{h_E^c(\mathcal{M})}.$$

Similarly,

$$\|z\|_{H_E^c(\mathcal{M})} \leq C\|z\|_{h_E^d(\mathcal{M})}, \quad \forall z \in h_E^d(\mathcal{M}).$$

Hence

$$\|y\|_{H_E^c(\mathcal{M})} \leq C \inf\{\|w\|_{h_E^d(\mathcal{M})} + \|z\|_{h_E^c(\mathcal{M})}\},$$

where the infimum runs over all decomposition  $y = w + z$  with  $w$  in  $h_E^c(\mathcal{M})$  and  $z$  in  $h_E^d(\mathcal{M})$ . So,  $h_E^c(\mathcal{M}) + h_E^d(\mathcal{M}) \subset H_E^c(\mathcal{M})$ . Using Theorem 2.3 and Lemma 2.4 we obtain that  $(h_E^c(\mathcal{M}) + h_E^d(\mathcal{M}))^* = h_{E^\times}^c(\mathcal{M}) \cap h_{E^\times}^d(\mathcal{M})$ . On the other hand,  $E$  is a separable symmetric Banach space on  $[0, 1]$  with  $1 < p_E \leq q_E < 2$ , so  $E$  is reflexive. Hence  $E^* = E^\times$  is separable. Applying Proposition 2.5 we deduce that  $H_E^c(\mathcal{M}) = h_E^c(\mathcal{M}) + h_E^d(\mathcal{M})$  with equivalent norms.

(ii) Passing to adjoint, we obtain  $H_E^r(\mathcal{M}) = h_E^r(\mathcal{M}) + h_E^d(\mathcal{M})$  with equivalent norms.  $\square$

By Proposition 2.5, 2.6 and Proposition 4.18 in [12], we obtain the following:

**Theorem 2.7.** *Let  $E$  be a separable symmetric Banach space on  $[0, 1]$  with  $1 < p_E \leq q_E < 2$ . Then we have*

$$L_E(\mathcal{M}) = h_E^c(\mathcal{M}) + h_E^d(\mathcal{M}) + h_E^r(\mathcal{M}).$$

**Remark 2.8.** Using Theorem 2.3 and Theorem 6.2 in [10] we can obtain the result of Theorem 2.7.

**Remark 2.9.** Should Theorem 2.7 be true whenever  $\mathcal{M}$  is semi-finite and  $E$  has the Fatou property. This was proved recently in [27] by Randrianantoanina and Wu via a very different method.

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